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RESEARCH ARTICLE

Equality of Relative Elementary and Relative Cohn Orbits

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Abstract

We define a special type of transformations called Cohn transformation and relative Cohn transformations. It is shown that the set of orbits of the action of relative Cohn transformation on all unimodular rows over a commutative ring is same as the set of orbits of the action of relative elementary transformation on unimodular rows.

Keywords: Unimodular rows, relative elementary transformation, relative Cohn transformation.
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Introduction

Let R be a commutative ring with 1. The study of the orbit space of unimodular rows of length r ≥ 3 was begun by Vaserstein in 1969. He showed that if R is a two dimensional ring then the orbit space $Um_3(R)/E_3(R)$ has an abelian Witt group structure. To prove this theorem, Vaserstein evolves the study of the action of the elementary group on an invertible alternating matrix.

The main result which is proved in section 3, Theorem 3.7 is obtained by observing the action of elementary matrices on an invertible alternating matrix. An **alternating matrix** φ is a skew-symmetric matrix (i.e. $\varphi^T = -\varphi$) with diagonal entries 0.

For instance, for $v = (a, b, c)$ and $w = (a', b', c')$, L.N. Vaserstein associates an alternating matrix

$$V(v, w) = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & c' & -b' \\ -b & -c' & 0 & a' \\ -c & b' & -a' & 0 \end{pmatrix}$$

The study of the action of elementary matrices on an invertible alternating matrix is directly connected to the action of the elementary group on the skew completable odd sized rows.

Materials and methods

2. Preliminaries: In this section we recall a few definitions, state some results and fix some notations which will be used throughout this study.

Definition 2.1 A row $v = (v_1, v_2, \dots, v_n) \in R^n$ is said to be unimodular if there are elements $w_1, w_2, \dots, w_n \in R$ such that $v_1w_1 + v_2w_2 + \dots + v_nw_n = 1$. $Um_n(R)$ will denote the set of all unimodular rows $v \in R^n$.

The vector $w = (w_1, w_2, \dots, w_n)$ is said to be related to v if the inner product $\langle v, w\rangle = 1$.

In the *n* x *n* matrices there are n^2 particular matrices that play a key role. These are called the **matrix units**, e_{ij}, $1 \le i, j \le n$ which are defined as follows: e_{ij} is the matrix whose ij-thentry is 1 and all other entries are 0.

The **General Linear group** $GL_n(R)$ is defined as the group of $n \times n$ invertible matrices with entries in R.

The **Special Linear group** is denoted by $SL_n(R)$ and is defined as $SL_n(R) = \{\alpha \in GL_n(R) : \det(\alpha) = 1\}$.

The group of elementary matrices $E_n(R)$ is a subgroup of $GL_n(R)$ generated by matrices of the form $E_{ij}(\lambda) = I_n + \lambda e_{ij}$, where $\lambda \in R, i \neq j$ and $e_{ij} \in M_n(R)$ is the matrix unit.

Following are some well-known properties of the elementary generators.

Lemma 2.2 *For* $\lambda, \mu \in R$,
(i) (Splitting Property) $E_{ij}(\lambda + \mu) = E_{ij}(\lambda)E_{ij}(\mu), 1 \le i \neq j \le n$
(ii) (Commutator Law) $[E_{ij}(\lambda), E_{jk}(\mu)] = E_{ik}(\lambda\mu), 1 \le i \neq j \neq k \le n$

In view of the Commutator Law, $E_n(R)$ is generated by $\{E_{1i}(\lambda), E_{i1}(\mu): 2 \le i \le n, \lambda, \mu \in R\}$. As R is commutative, $E_{ij}(\lambda)$, $i \neq j, \lambda \in R$, is invertible with inverse $E_{ij}(-\lambda)$. In fact, $E_{ij}(\lambda)$ belongs to $SL_n(R)$. Hence,
$E_n(R) \subseteq SL_n(R) \subseteq GL_n(R)$.

If M is an $n \times n$ matrix, left multiplication of M by $E_{ij}(\lambda)$ corresponds to adding a λ-multiple of the j^{th} row of M to the i^{th} row of M, and the right multiplication of M by

$E_{ij}(\lambda)$ corresponds to adding a λ - multiple of the i^{th} column of M to the j^{th} column of M. These are called **elementary transformations**.

We say that two vectors v and u in R^n are said to be in the same **elementary orbit** if there exist an $\varepsilon \in E_n(R)$ such that $v\varepsilon = u$.

Proposition 2.3 (Suslin, 1977)
Let $v = (v_1, \dots, v_r) \in Um_r(R)$. Let $\varphi : R^r \to R$ be a map such that $\varphi(e_i) = v_i$, where $\{e_1, \dots, e_r\}$ is the standard basis of R^r. Then kernel of φ is generated by the set $\{v_j e_i - v_i e_j : 1 \le i < j \le r\}$.

Proof: Let $u = (u_1, \dots, u_r) \in R^r$ be such that $\langle u, v \rangle = u \cdot v^T = 1$. If $w = (w_1, \dots, w_r) \in ker\ \varphi$, then $w = \sum_{i=1}^{r} w_i e_i = \sum_{i=1}^{r} w_i (e_i - v_i u)$. So $\{e_1 - v_1 u, \dots, e_r - v_r u\}$ generates the kernel of φ. But $e_i - v_i u = \sum_{k=1, k\ne i}^{r} u_k (v_k e_i - v_i e_k)$. Thus the kernel of φ is generated by the set $\{v_j e_i - v_i e_j : 1 \le i < j \le r\}$.

Definition 2.4 Let $v = (a_1, a_2, \dots, a_r)$, $w = (b_1, b_2, \dots, b_r) \in R^r$ with $\langle v, w \rangle = v_1 w_1 + v_2 w_2 + \cdots + v_r w_r = 1$.
We say that the vector
$v^* = vC_{ij}(\lambda) = (a_1, \dots, a_i + \lambda b_j, \dots, a_j - \lambda b_i, \dots, a_r)$,
for $0 \le i \ne j \le r$, is a **Cohn transform** of *v* w.r.t. the vector *w*.

One could write $C_{ij}(w, \lambda)$ if one wish to specify *w*, but we generally refrain from doing so, as the row *w* being considered is clear from the context.

We shall say that a row v^* is in the **Cohn orbit** of *v* if there is a row w^* with $\langle v^*, w^* \rangle = 1$, and a sequence of pairs, starting with $(v_0, w_0) = (v, w)$, and ending with $(v_n, w_n) = (v^*, w^*)$, such that, for $i \ge 0$, the pairs (v_{i+1}, w_{i+1}) has either v_{i+1} as a Cohn transform of v_i w.r.t. w_i, and $w_{i+1} = w_i$; or w_{i+1} as a Cohn transform of w_i w.r.t. v_i, and $v_{i+1} = v_i$:

$$(v, w) = (v_0, w_0) \to (v_1, w_1) \to \cdots \to (v_n, w_n) = (v^*, w^*).$$

Results and discussion

3. Equality of Relative Cohn and Relative Elementary Orbits: In this section we start with the definition of relative elementary group and relative Cohn transformation:

Definition 3.1 Let I be an ideal of R. The group $E_n(I)$ is the subgroup of $E_n(R)$ generated by the elements $E_{ij}(x), x \in I, 1 \le i \ne j \le n$. The relative elementary group $E_n(R, I)$ is the smallest normal subgroup of $E_n(R)$ containing $E_n(I)$.

Also $E_n(R, I)$ is generated by the elements $E_{ij}(a)E_{ji}(x)E_{ij}(-a)$, with $a \in R$, $x \in I$ and $1 \le i \ne j \le n$, provided $n \ge 3$.

Definition 3.2 Let I be an ideal of R. A row is said to be relative unimodular w.r.t. I if it is unimodular and congruent to $e_1 = (1, 0, \dots, 0)$modulo I. $Um_n(R, I)$ denote the set of all relative unimodular rows w.r.t. I of length n. If $I = R$, then $Um_n(R, I)$ is $Um_n(R)$.

Definition 3.3 Let I be an ideal of R.
Let $v = (a_1, a_2, \dots, a_n)$, $w = (b_1, b_2, \dots, b_n) \in R^n$ with $\langle v, w \rangle = 1$ and v is congruent to e_1 modulo I. We say that the vector

$v^* = vC_{1i}(\lambda) = (a_1 + \lambda b_i, \dots, a_i - \lambda b_1, \dots, a_n)$, for $2 \le i \le n, \lambda \in I$

is a Relative Cohn transform of v w.r.t. the vector w. We denote it with $RC_{ij}(\lambda)$. Thus

$$RC_{ij}(\lambda) = \begin{cases} C_{ij}(\lambda)\ if\ i = 1\ or\ j = 1\ and\ \lambda \in I \\ C_{ij}(\lambda)\ if\ i \ne 1, j \ne 1\ and\ \lambda \in R \end{cases}$$

Lemma 3.4 (Suslin, 1977; Corollary 1.2)
Let $n \ge 3$ and I be an ideal of R. Let $v \in R^n$ and $w \in I^n$ be such that $\langle w, v \rangle = 0$. If $w_i = 0$, for some $1 \le i \le n$, then $I_n + v^t w \in E_n(R, I)$.

Lemma 3.5 Let $n \ge 3$ and I be an ideal of R. Let $v \in Um_n(R)$ and $w \in I^n$ such that $\langle w, v \rangle = 0$. Then $I_n + v^t w \in E_n(R, I)$.

Proof: Let $v = (v_1, v_2, \dots, v_n) \in R^n$ and $w = (w_1, w_2, \dots, w_n) \in I^n$. Let $u = (u_1, u_2, \dots, u_n) \in R^n$ such that $v_1 u_1 + v_2 u_2 + \cdots + v_n u_n = 1$. Using Lemma 2.3, we get,

$$\begin{aligned} w &= \sum_{i=1}^{n} w_i e_i = \sum_{i \ne j} v_j (w_i u_j - w_j u_i) e_i \\ &= \sum_{i<j} (w_i u_j - w_j u_i)(v_j e_i - v_i e_j) \\ &= \sum_{i<j} a_{ij} (v_j e_i - v_i e_j) \end{aligned}$$

where $a_{ij} \in I$. Now

$$\begin{aligned} I_n + v^t w &= I_n + \sum_{i<j} a_{ij} v^t (v_j e_i - v_i e_j) \\ &= \prod_{i<j} (I_n + a_{ij} v^t (v_j e_i - v_i e_j)) \end{aligned}$$

By Lemma 3.4, each term appeared in the above product is in $E_n(R, I)$. Hence the result:

Lemma 3.6 Let $v_1, v_2, w \in R^n, n \geq 3$.
If $v_1, v_2 \in Um_n(R, I)$ and $\langle v_1, w \rangle = 1 = \langle v_2, w \rangle$, then there exists $\varepsilon \in E_n(R, I)$ such that $v_1 = v_2\varepsilon$, and with $w\varepsilon^{T^{-1}} = w$.

Proof: Since $\langle v_1, w \rangle = 1 = \langle v_2, w \rangle$, we have $\langle v_2 - v_1, w \rangle = 0 = \langle w, v_2 - v_1 \rangle$. By Lemma 3.5, $\varepsilon = I_r + (v_2 - v_1)^T w \in E_r(R, I)$. Clearly $v_1\varepsilon = v_2$, and $w\varepsilon^{T^{-1}} = w\big(I_r + (v_1 - v_2)\big) = w$.

Theorem 3.7 The relative elementary orbit $vE_n(R, I)$ of $v \in Um_n(R, I)$ coincide with the relative Cohn orbit of v for $n \geq 3$.

Proof: Suppose v^* is in the relative Cohn orbit of v. Then there is a row w^* with $\langle v^*, w^* \rangle = 1$, and a sequence of pairs, starting with $(v_0, w_0) = (v, w)$, and ending with $(v_n, w_n) = (v^*, w^*)$, such that, for $i \geq 1$, the pairs (v_{i+1}, w_{i+1}) has either v_{i+1} as a relative Cohn transform of v_i w.r.t. w_i, and $w_{i+1} = w_i$ or w_{i+1} as a relative Cohn transform of w_i w.r.t. v_i, and $v_{i+1} = v_i$:

$$(v, w) = (v_0, w_0) \to (v_1, w_1) \to \cdots \to (v_n, w_n) = (v^*, w^*).$$

By Lemma 3.6, v_0, v_1 are in the same relatively elementary orbit and w_0, w_1 are in the same relatively elementary orbit. It follows inductively that $v^* \in vE_n(R, I)$, i.e. relative Cohn transforms lie in the same relative elementary orbit.

Conversely, any relative elementary transformation of a row can be obtained by means of a sequence of relative Cohn transforms with respect to some suitable rows. Let $v = (a_1, a_2, \ldots, a_n)$ and $w = (b_1, b_2, \ldots, b_n)$. Then for $1 < i \neq j < n, \lambda \in R, x \in I$,

$$\begin{aligned}
v_1 &= vRC_{1j}(\lambda) = \big(a_1 + \lambda b_j, a_2, \ldots, a_j - \lambda b_1, \ldots, a_n\big)\\
w_1 &= w\\
v_2 &= v_1\\
w_2 &= w_1RC_{ij}(1)\\
&= \big(b_1, \ldots, b_i + a_j - \lambda b_1, \ldots, b_j - a_i, \ldots, b_n\big)\\
v_3 &= v_2RC_{1j}(-\lambda)\\
&= \big(a_1 + \lambda b_j - \lambda(b_j - a_i), a_2, \ldots, a_j - \lambda b_1 + \lambda b_1, \ldots, a_n\big)\\
&= (a_1 + \lambda a_i, a_2, \ldots, a_n) = vE_{i1}(\lambda)\\
w_3 &= w_2\\
v_4 &= v_3\\
w_4 &= w_3RC_{ij}(-1)\\
&= \big(b_1, \ldots, b_i + a_j - \lambda b_1 - a_j, \ldots, b_j - a_i + a_i, \ldots, b_n\big)\\
&= (b_1, \ldots, b_i - \lambda b_1, \ldots, b_n) = w\,E_{1i}(-\lambda)\\
v_5 &= v_4RC_{1j}(x)\\
&= \big(a_1 + \lambda a_i, \ldots, a_i + xb_j, \ldots, a_j - xb_i + \lambda x\, b_1, \ldots, a_n\big)\\
w_5 &= w_4\\
v_6 &= v_5\\
w_6 &= w_5RC_{ij}(1)\\
&= \big(b_1 + a_j - xb_i + \lambda x\, b_1, \ldots, b_i - \lambda b_1, \ldots, b_j - a_1 - \lambda a_i, \ldots, b_n\big)\\
v_7 &= v_6RC_{ij}(-1)\\
&= (a_1 + \lambda v_i, \ldots, v_i + xv_1 + \lambda x\, v_i, \ldots, v_n)\\
&= vE_{i1}(\lambda)E_{1i}(x)\\
w_7 &= w_6\\
v_8 &= v_7\\
w_8 &= w_7RC_{ij}(-1)\\
&= (b_1 - xb_i + \lambda x\, b_1, \ldots, b_i - \lambda b_1, \ldots, b_n)\\
&= vE_{1i}(-\lambda)E_{i1}(-x)\\
v_9 &= v_8RC_{1j}(-\lambda)\\
&= \big(a_1 + \lambda a_i - \lambda b_j, \ldots, a_i + xa_1 + \lambda x\, a_i, \ldots, a_j + \lambda(b_1 - xb_i + \lambda x\, b_i), \ldots, a_n\big)\\
w_9 &= w_8\\
v_{10} &= v_9\\
w_{10} &= w_9RC_{ij}(-1)\\
&= \big(b_1 - xb_i + \lambda x\, b_1, \ldots, b_i - \lambda b_1 - \big(a_j - \lambda b_1 - \lambda x\, b_i - \lambda^2 x\, b_i\big), \ldots, b_j + (a_i + xa_1 + \lambda x\, a_i), \ldots, b_n\big)\\
v_{11} &= v_{10}RC_{1j}(\lambda)\\
&= (a_1 + \lambda x\, a_1 + \lambda^2 x\, a_i, \ldots, a_i + xa_1 + \lambda x\, a_i, \ldots, a_n)\\
&= vE_{i1}(\lambda)E_{1i}(x)E_{i1}(-\lambda)\\
w_{11} &= w_{10}\\
v_{12} &= v_{11}\\
w_{12} &= w_{11}RC_{ij}(1)\\
&= (b_1 - xb_i + \lambda x\, b_1, \ldots, b_i + \lambda x\, b_i + \lambda^2 x\, b_i, \ldots, b_n)\\
&= wE_{1i}(-\lambda)E_{i1}(-x)E_{1i}(\lambda)
\end{aligned}$$

Thus, we have proved that a relatively elementary transformation of a row can be obtained by means of a sequence of relative Cohn transformation w.r.t. some suitable rows. This completes the proof.

Conclusion

We established that the set of orbits of the action of relative Cohn transformation on all unimodular rows over a commutative ring is same as the set of orbits of the action of relative elementary transformation on unimodular rows.

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